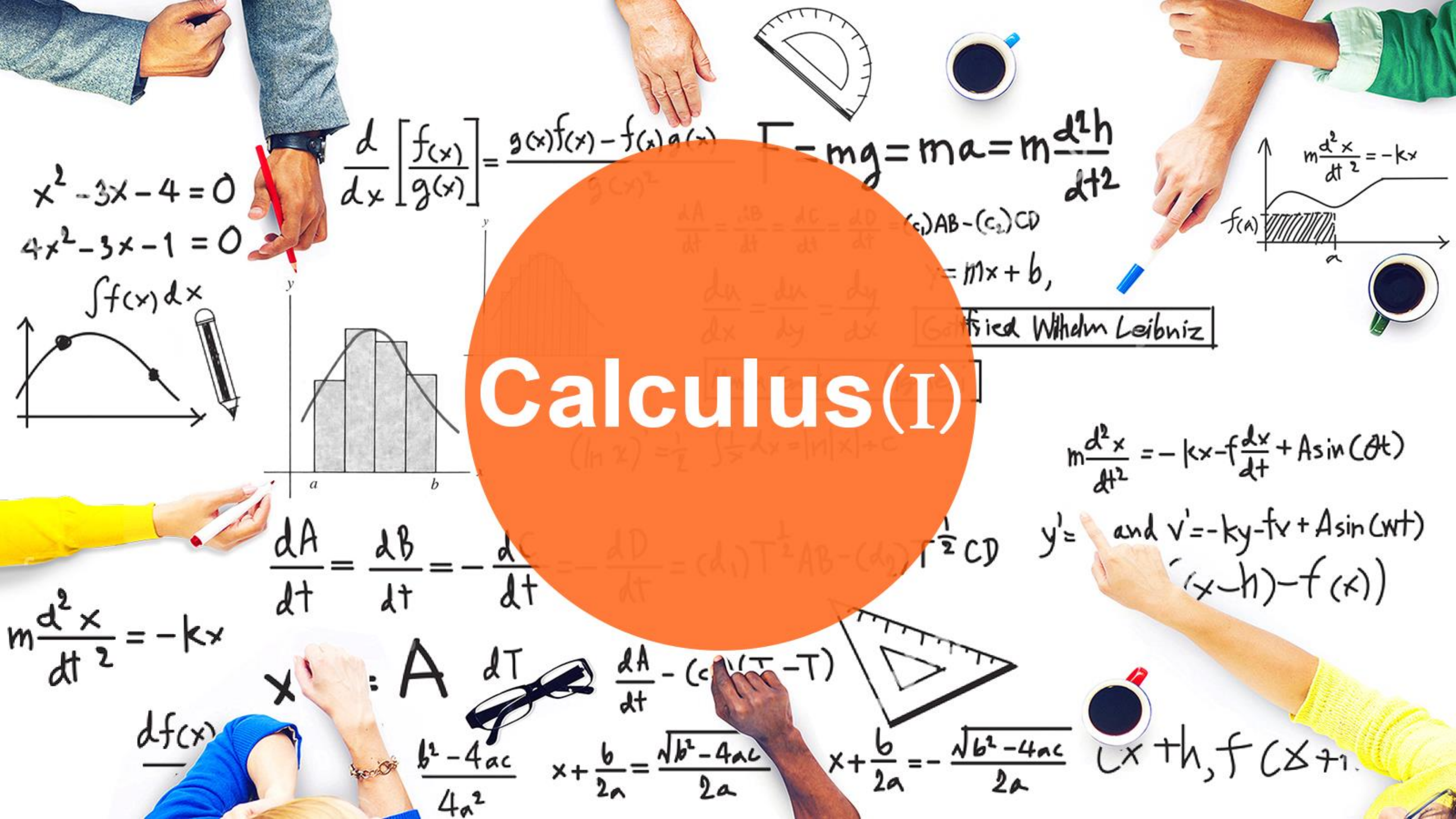
Calculus(I)

Higher-Order Derivatives

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Problem Introduction

Acceleration of variable speed linear motion?

Let $s = s(t)$, so instantaneous velocity $v(t) = s'(t)$

$\because a$ is the rate: the change rate of $v(t)$ to t

$$\therefore a(t) = v'(t) = [s'(t)]' = s''(t).$$

The physical meaning of the second derivative.

Definition of Higher-Order Derivatives

$$f''(x) = (f'(x))' = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$$

The second derivative:

$$f''(x),\ y'',\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \text{ or } \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}.$$

Definition of Higher-Order Derivatives

the third derivative:	$f'''(x),\ y''',\ \dfrac{d^3y}{dx^3}.$
the fourth derivative:	$f^{(4)}(x),\ y^{(4)},\ \dfrac{d^4y}{dx^4}.$
……	……
the *n*-th derivative:	$f^{(n)}(x),\ y^{(n)},\ \dfrac{d^ny}{dx^n}$ or $\dfrac{d^nf(x)}{dx^n}.$

the *n*-th derivative is called higher-order derivative **($n \geq 2$).**

$f(x)$ is called zero order derivative $(n=0)$,

$f'(x)$ is called the first derivative $(n=1)$.

Example 1

If $y = \sin 2x$, find $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, and $\frac{d^{12}y}{dx^{12}}$.

$$\frac{dy}{dx} = 2\cos 2x$$

$$\frac{d^2y}{dx^2} = -2^2 \sin 2x$$

$$\frac{d^3y}{dx^3} = -2^3 \cos 2x$$

$$\frac{d^4y}{dx^4} = 2^4 \sin 2x$$

$$\frac{d^5y}{dx^5} = 2^5 \cos 2x$$

$$\cdots\cdots$$

$$\frac{d^{12}y}{dx^{12}} = 2^{12} \sin 2x.$$

Example 2

If $y = \arctan x$, find $y''|_{x=0}, y'''|_{x=0}$.

$$y' = \frac{1}{1+x^2}$$

$$y'' = \left(\frac{1}{1+x^2}\right)' = \frac{-2x}{(1+x^2)^2}$$

$$y''' = \left[\frac{-2x}{(1+x^2)^2}\right]' = \frac{2(3x^2-1)}{(1+x^2)^3}$$

$$\therefore y''|_{x=0} = \left.\frac{-2x}{(1+x^2)^2}\right|_{x=0} = 0; \quad y'''|_{x=0} = \left.\frac{2(3x^2-1)}{(1+x^2)^3}\right|_{x=0} = -2.$$

Example 3

If $y = e^x$, find $y^{(n)}$.

$$y' = e^x,$$

$$y'' = e^x,$$

$$y''' = e^x,$$

$$\cdots\cdots$$

$$\therefore (e^x)^{(n)} = e^x.$$

Example 4

If $y = \ln(1+x)\,(x > -1)$, find $y^{(n)}$.

$$y' = \frac{1}{1+x}$$

$$y'' = -\frac{1}{(1+x)^2}$$

$$y''' = \frac{2!}{(1+x)^3}$$

$$y^{(4)} = -\frac{3!}{(1+x)^4}$$

$$\cdots\cdots$$

$$\therefore y^{(n)} = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n} \qquad (n \geq 1,\ 0! = 1)$$

Example 5

If $y = \sin x$, find $y^{(n)}$.

$$y' = \cos x = \sin(x + \frac{\pi}{2})$$

$$y'' = \cos(x + \frac{\pi}{2}) = \sin(x + \frac{\pi}{2} + \frac{\pi}{2}) = \sin(x + 2 \cdot \frac{\pi}{2})$$

$$y''' = \cos(x + 2 \cdot \frac{\pi}{2}) = \sin(x + 3 \cdot \frac{\pi}{2})$$

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$$\therefore y^{(n)} = \sin(x + n \cdot \frac{\pi}{2})$$ so $$(\sin x)^{(n)} = \sin(x + n \cdot \frac{\pi}{2})$$

In the similar way $$(\cos x)^{(n)} = \cos(x + n \cdot \frac{\pi}{2})$$

Summary of Higher-Order Derivatives

The n-th order derivative of some special functions

(1) $(a^x)^{(n)} = a^x \cdot \ln^n a \quad (a > 0) \qquad (e^x)^{(n)} = e^x$

(2) $(\sin kx)^{(n)} = k^n \sin(kx + n \cdot \frac{\pi}{2})$

(3) $(\cos kx)^{(n)} = k^n \cos(kx + n \cdot \frac{\pi}{2})$

(4) $(x^\alpha)^{(n)} = \alpha(\alpha - 1)\cdots(\alpha - n + 1)x^{\alpha - n}$

(5) $(\ln x)^{(n)} = (-1)^{n-1} \frac{(n-1)!}{x^n} \qquad \left(\frac{1}{x}\right)^{(n)} = (-1)^n \frac{n!}{x^{n+1}}$

Questions and Answers

If $y = x^{\alpha}(\alpha \in \mathrm{R})$, find $y^{(n)}$.

$$y' = \alpha x^{\alpha-1}$$

$$y'' = (\alpha x^{\alpha-1})' = \alpha(\alpha-1)x^{\alpha-2}$$

$$y''' = (\alpha(\alpha-1)x^{\alpha-2})' = \alpha(\alpha-1)(\alpha-2)x^{\alpha-3}$$

$$\cdots\cdots$$

$$\therefore y^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n} \qquad (n \geq 1)$$

If α is a natural number n, so

$$\boldsymbol{y^{(n)} = (x^n)^{(n)} = n!,\quad y^{(n+1)} = (n!)' = 0.}$$

Questions and Answers

If $y=\dfrac{1}{x^2-3x+2}$, find $y^{(n)}$.

$$y=\frac{1}{x^2-3x+2}=\frac{1}{x-2}-\frac{1}{x-1}$$

$$y'=\frac{-1}{(x-2)^2}-\frac{-1}{(x-1)^2}=(-1)\frac{1}{(x-2)^2}-(-1)\frac{1}{(x-1)^2}$$

$$y''=\frac{(-1)(-2)}{(x-2)^3}-\frac{(-1)(-2)}{(x-1)^3}=(-1)^2\frac{1\cdot 2}{(x-2)^3}-(-1)^2\frac{1\cdot 2}{(x-1)^3}$$

$$\cdots\cdots$$

$$\therefore\ y^{(n)}=(-1)^n\frac{n!}{(x-2)^{n+1}}-(-1)^n\frac{n!}{(x-1)^{n+1}}$$

$$=(-1)^n n!\left[\frac{1}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right]$$

Questions and Answers

If $y=\sin^4 x+\cos^4 x$, find $y^{(n)}$.

Hint: If the derivation is very complex,

it isn't easy to find out the rules, try to change the formula.

$$(\cos x)^{(n)}=\cos(x+n\cdot\frac{\pi}{2})$$

$$y=\sin^4 x+\cos^4 x$$

$$=(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x$$

$$=1-\frac{1}{2}\sin^2 2x \quad =1-\frac{1}{2}\left(\frac{1-\cos 4x}{2}\right) \quad =\frac{3}{4}+\frac{1}{4}\cos 4x$$

$$\therefore y^{(n)}=\frac{1}{4}\cdot 4^n\cos\left(4x+n\cdot\frac{\pi}{2}\right)$$

Questions and Answers

If $y=(x+2)(2x+3)^2(3x+4)^3$, find $y^{(6)}$.

Idea: This function is the 6-degree polynomial, and try to find the 6th-order derivative.

Because $y = x(2x)^2(3x)^3 + p_5(x)$

$$=108x^6 + p_5(x)$$

And $p_5(x)$ is the 5th-degree polynomial of x.

So $y^{(6)} = 108 \cdot 6!$.

Higher-Order Derivatives